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A unified unsteady lifting-line theory

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A lifting-line theory is developed for wings of large aspect ratio oscillating in an inviscid fluid. The theory is unified in the sense that the wing may be curved or inclined to the flow, and the asymptotic expansion is uniformly valid with respect to the frequency. The method is based on the integral equation formulation of the problem. The technique, pioneered by Kida & Miyai (1978), consists of asymptotically solving the Fredholm equation of the first kind which links the unknown pressure jump and the normal velocity imposed on the wing. Use of the finite-part integral theory introduced by Hadamard (1932) and of a technique developed in Guermond (1987, 1988, 1990) yields an asymptotic expansion of the surface integral in terms of the inverse of the aspect ratio. At each approximation order, the problem reduces to a classical two-dimensional integral equation, whose unknown is the pressure jump, and whose right-hand side depends only on the previous approximation orders of the solution. The first finite-span correction is explicitly calculated. An extensive numerical study is carried out, and comparisons with published results are made.

1. Introduction

As computer capacities are progressing at a spectacular pace, the range of application of computational fluid dynamics is becoming broader. Complex flows of real fluids can now be quite accurately simulated at reasonable numerical costs. Nonetheless, numerical simulations of flows do not give clear qualitative insight into the physical mechanisms which take place. This perspective, highly desirable for designers, may be attained by asymptotic modelling.

The lifting-line model, originally proposed by Prandtl (1921) and mathematically justified by Van Dyke (1964), is one of the most popular asymptotic models in aerodynamics. It is based on two hypotheses. The first one consists of assuming that the ratio of the span lengthscale, B, of a lifting wing to its chord lengthscale, C, is much greater than one ($1 \ll B/C$). This ratio, called the aspect ratio, is denoted by A. The second hypothesis is that the variation of the flow along the span occurs on the long lengthscale B.

The lifting-line model is widely taught and used to illustrate the major feature of the flow field produced by a finite lifting wing at subsonic speeds. With the help of their simplified model, Prandtl and Van Dyke taught scores of students that the higher the aspect ratio of the wing the higher its efficiency. They showed that the ratio of the induced drag to the total lift is proportional to $1/A$. Whereas Prandtl's method involved solving an integral equation, Van Dyke recognized that the problem could be considered as a singular perturbation, and he consequently

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introduced the matched asymptotic expansion technique (MAE) into the solution of the problem. The success of his approach prompted many other researchers to use it for devising extensions of the original model which was restricted to unswept wings operating in steady flow.

The restrictions on the wing geometry were removed by Cheng (1978) and Cheng & Murillo (1984) using the MAE approach. The mechanism that is responsible for shifting the loading from the wing centre to the wing tips when the wing is curved or swept backward was clearly illuminated by the lifting-line theory. It was shown that the backward inclination of the upstream edge of the trailing vortices produces logarithmically singular upwash in the vicinity of the wing tips, whereas the curvature of the bound vortices generates strong downwash in the vicinity of the wing root. The phenomenon is reversed when the wing is curved or swept forward. As reported in Cheng & Murillo (1984), this mechanism 'explains why a swept-back wing has a lesser margin to tip stall than an unswept or swept-forward wing'.

The asymptotic theory has also been extended to unsteady flow. Most authors have considered harmonic perturbations of the boundary conditions. To characterize the periodic structure of the wake, it is convenient to introduce the wavelength $\lambda = 2\pi U/\omega$, where U is the free-stream velocity and ω is the radian frequency of perturbations. Under the hypothesis of a high-aspect-ratio wing ($C \ll B$), Cheng (1976) has identified five frequency domains: the very low frequencies ($B \ll \lambda$), the low frequencies ($B = O(\lambda)$), the intermediate frequencies ($C \ll \lambda \ll B$), the high frequencies ($C = O(\lambda)$), and the very high frequencies ($\lambda \ll C$).

One major result, brought out by the lifting-line theory of unswept straight wings, consists of a self-averaging effect when the frequency is high or very high. This result, originally formulated in a simplified form by James (1975) and by Cheng (1976), has been thoroughly studied by Guiraud & Slama (1981). They showed that, in the high- and very-high-frequency domain, the three-dimensional corrections due to the finite aspect ratio are of $O(\log(A)/A^2)$ instead of being of $O(1/A)$ as in the steady case. They explained the phenomenon by a double scale mechanism. Indeed, λ is the lengthscale of the unsteady perturbations induced by the wing. These perturbations are convected in the wake far downstream in the outer domain at distances of order B and greater. Hence, in a small sheath surrounding the wake, the two lengthscales λ and B must be accounted for, whereas outside the sheath only the lengthscale B need be considered. In the high-frequency domain, the persistent small lengthscale λ is solely responsible for the self-averaging phenomenon. It was proposed in Cheng (1976) that this phenomenon was also likely to occur on swept and curved wings of high aspect ratio. Actually, it is shown here that a term, proportional to the local curvature and the local sweep angle, is still of $O(1/A)$ in this frequency domain. It finally becomes of $O(\log(A)/A^2)$ in the very-high-frequency domain.

Another interesting phenomenon concerns the shape of the induced downwash with respect to the chordwise location. In early papers by James (1975) and Van Holten (1976) on unsteady straight lifting-line theory, it was assumed that, like in the steady case, the induced downwash was constant over the chord for the entire range of frequencies. Ahmadi & Widnall (1985) showed that this assumption was asymptotically correct only for very low frequencies ($B \ll \lambda$), and they proved that in the low-frequency domain ($B = O(\lambda)$) the downwash had a sinusoidal dependence on the chordwise variable. Their work was focused on these two frequency domains. It is shown, in further developments of this paper, that Ahmadi & Widnall's second conclusion is actually true for the entire range of frequencies. For straight upswept wings, at the first order of approximation, the downwash behaves like a sine of wavelength λ. This shape, however, is altered if sweep and curvature are taken into account.

Because of the difficulties involved, previous work published on unsteady lifting-line theory either covers a small frequency domain or does not take into account sweep and curvature effects. The cause of non-uniformity in the frequency domain has been analysed by Sclavounos (1987). It comes from the desire to asymptotically relate the wavelength λ of perturbations to either the span lengthscale B or the chord lengthscale C. Such a comparison automatically restricts subsequent considerations to a narrow frequency domain. The restrictions in terms of sweep and curvature of previous MAE approaches seem to come from the use of curvilinear coordinates in the inner domain. These coordinate systems render the inner problem somewhat untractable. It is the purpose of the present work to present a lifting-line theory which is valid over the entire range of frequencies, whatever the shape of the wing.

The present paper mainly consists of the generalization of Guermond (1990) to unsteady flows. The solution is based on the integral equation formulation of the problem. The technique, pioneered by Kida & Miyai (1978), consists of asymptotically solving the Fredholm equation of the first kind which links the unknown pressure jump and the normal velocity imposed on the wing. An asymptotic expansion of the surface integral with respect to A is found using the finite-part integral theory introduced by Hadamard (1932) and a particular technique developed in Guermond (1987, 1988, 1990). At each approximation order the problem reduces to a classical, two-dimensional, integral equation, whose unknown is the pressure jump, and whose right-hand side depends only on the previous approximation orders of the solution. The finite-span corrections of the pressure jump are explicitly calculated up to $O(1/A)$. An extensive numerical study has been carried out and comparisons with published results are made.

2. Formulation of the problem

2.1. *General assumptions*

Consider a Cartesian coordinate system $(OXYZ)$ and a uniform, incompressible, irrotational stream of an inviscid fluid with density ρ. The velocity U of the free stream is directed along the OX-axis. The OZ-axis and the plane $Z = 0$ are referred to as the vertical positive direction and the reference plane. Consider a thin, almost-planar wing of large aspect ratio placed in the above-mentioned flow. The wing is sufficiently close to the reference plane that the classical linearization process applies. The projection of the wing surface and its trailing vortex sheet onto the reference plane are denoted by S and Σ.

The spanwise mean geometry of the wing is modelled by a smooth line L (see figure 1) whose equation is

$$X_0 = Bx_0(Y) \quad \text{for all}\ M_0(X_0, Y)\ \text{on}\ L. \tag{1}$$

This line may not necessarily be on S but must be at a distance of S, in the topological sense, of order C:

$$\sup_{\substack{M_0 \in L \\ P \in S \cap M_0 X}} |M_0 P| = O(C). \tag{2}$$

For the sake of simplification in further developments, L is taken as a reference line in the streamwise direction, and the following new set of non-dimensional coordinates is defined:

$$X = Cx + Bx_0(y), \quad Y = By. \tag{3}$$

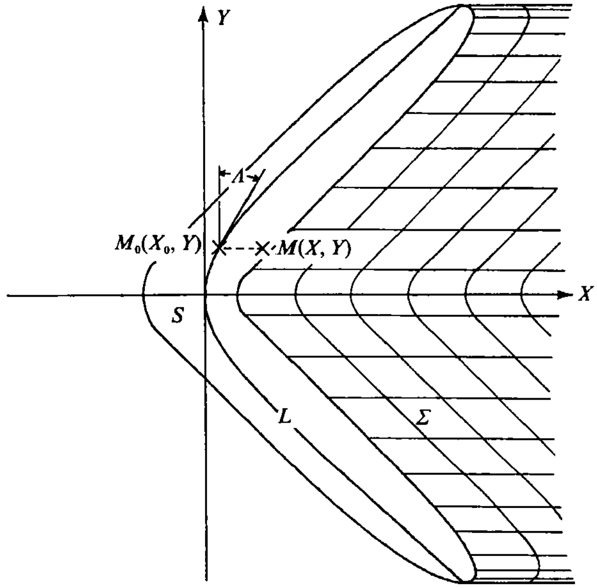

FIGURE 1. Definition of the Cartesian coordinate system and geometric parameters.

The line L is assumed to be smooth in the sense that the function $x_0(y)$ is of order one and has as many derivatives with respect to y as needed, these derivatives also being of unit order. Using the non-dimensional coordinates, the equation of the wing planform S is given by

$$c_1(y) \leqslant x \leqslant c_t(y) \quad \text{for} \quad -1 \leqslant y \leqslant +1, \tag{4}$$

where $c_1(y)$ and $c_t(y)$ are the abscissae of the leading edge and trailing edge respectively. Owing to the condition (2), x, $c_1(y)$, and $c_t(y)$ are of order one. The geometry of the wing is assumed to be smooth in the sense that functions $c_1(y)$ and $c_t(y)$ have as many derivatives with respect to y of order one as needed. The local non-dimensional chord length $c(y)$ is given by the difference between $c_t(y)$ and $c_1(y)$.

The wing is assumed to undergo small, time-harmonic, vertical deformations or displacements, whose amplitude may vary along the span on the lengthscale B. The deformations, through the linearization process, are equivalent to small periodic, vertical disturbances of the incident flow. The displacements are due to heave and pitch motions. From now on, complex notation is adopted, and is implicitly understood that only the real part of the product of all complex quantities with the time factor $e^{i\omega t}$ is of importance. Under the linearization hypothesis, it is not necessary to make a distinction between deformations and displacements; only the vertical component of the velocity of these perturbations is of interest. The complex amplitude of the velocity is defined by $W(X, Y)$, and w is the non-dimensional ratio of W to U.

2.2. *The integral-equation formulation*

The chosen hypotheses imply that a perturbation velocity potential Φ exists. Defining the acceleration potential $\Psi = -P/\rho$, where P is the perturbation pressure, a simple relationship between Φ and Ψ is expressed by the linearized form of the Bernoulli law:

$$\Psi = (i\omega + U\partial/\partial X)\,\Phi. \tag{5}$$

Let $\mathscr{B}$ be the differential operator $(\mathrm{i}\omega + U\partial/\partial X)$. Provided Ψ vanishes at infinity, the inverse operator is found after some analysis:

$$\Phi = \mathscr{B}^{-1}(\Psi) = \frac{1}{U}\int_{-\infty}^{0} \mathrm{e}^{\frac{\mathrm{i}\omega V}{U}}\, \Psi(X+V, Y, Z)\,\mathrm{d}V. \tag{6}$$

A formulation in terms of the acceleration potential Ψ is sought, because Ψ is continuous everywhere except across the wing and the pressure jump is directly obtained from it. The PDE system satisfied by Ψ is classical and can be put into the form

$$\left.\begin{aligned} &\nabla^2\Psi = 0, \\ &\partial\Psi/\partial Z = \mathscr{B}(W(X,Y)) \quad \text{on} \quad S, \\ &[\![\Psi]\!] = 0 \quad \text{at the trailing edge}, \\ &\Psi \overset{\infty}{\to} 0. \end{aligned}\right\} \tag{7}$$

Here, $[\![\Psi]\!]$ is the jump of Ψ across S; it is equal to $\Psi_{\mathrm{u}} - \Psi_{\mathrm{l}}$, where subscripts u and l denote the upper and the lower sides of S respectively. The system (7) can be solved by means of the Green's representation theorem, and for all points $M(X_M, Y_M, Z_M)$ belonging to the flow domain the acceleration potential can be cast in the form

$$\Psi(M) = -\iint_{P\in S} [\![\Psi]\!] \frac{\partial}{\partial Z_P} G(M,P)\,\mathrm{d}S_P, \tag{8}$$

where $P(X_P, Y_P, 0)$ is any point on S, and $G(M,P)$ is the elementary Green function:

$$G(M,P) = -\frac{1}{4\pi|MP|}. \tag{9}$$

The tangency condition is imposed by applying the operator $\mathscr{B}^{-1}\circ\partial/\partial Z_M$ to Ψ and taking the limit as Z_M tends to zero. The following result is obtained:

$$W(X_M, Y_M) = \frac{1}{4\pi U}\mathrm{FP}\int_{-\infty}^{0} \mathrm{e}^{\frac{\mathrm{i}\omega V}{U}}\, \mathrm{FP}\iint_S \frac{[\![\Psi]\!]\,\mathrm{d}X_P\,\mathrm{d}Y_P}{[(X_M - X_P + V)^2 + (Y_M - Y_P)^2]^{\frac{3}{2}}}\,\mathrm{d}V. \tag{10}$$

It is important to note that the limiting process $(Z_M \to 0)$ implies that the integrals are defined in the finite-part sense (FP) as introduced by Hadamard (1932). A thorough review of Hadamard's theory may be found in Lavoine (1959, 1963), and some of the aspects pertaining to the present problem are summarized in Guermond (1990).

At this stage, the problem consists of inverting a Fredholm equation of the first kind. Following previous successful asymptotic treatments of this problem by Kida & Miyai (1978) and Guermond (1987, 1990), an asymptotic expansion of the integral, with respect to A, is sought.

3. Asymptotic formulation of the integral equation

3.1. *The non-dimensional equation*

In order to facilitate the asymptotic sorting of terms, dimensional variables X_M, Y_M, X_P, and Y_P are replaced by the non-dimensional variables x, y, ξ, and η as introduced in (3). The non-dimensional quantities associated with Φ, Ψ and V are defined by

$$\Phi = UC\phi, \quad \Psi = U^2\psi, \quad V = Bv. \tag{11}$$

As the two lengthscales C and B are independent, it is necessary to define two reduced frequencies, k and ν, based on them:

$$k = \omega C/U, \quad \nu = \omega B/U. \tag{12}$$

Note that ν is equal to kA. In the following, k and ν will be referred to as the chord reduced frequency and the span reduced frequency respectively. In further developments, no assumption is made on the order of magnitude of these reduced frequencies. As a result, the present theory is valid whatever the frequency. Furthermore, in order to separate the quasi-steady effects from the purely unsteady ones, an integration by parts of the right-hand side of (10) with respect to v is performed. After some simple calculation, (10) can be rewritten as:

$$\begin{aligned} w(x,y) = {} & \frac{A^{-1}}{4\pi}\mathrm{FP}\iint_S \frac{[\![\psi]\!]}{(y-\eta)^2}\left[1+\frac{A^{-1}(x-\xi)+x_0(y)-x_0(\eta)}{[(A^{-1}(x-\xi)+x_0(y)-x_0(\eta))^2+(y-\eta)^2]^{\frac{1}{2}}}\right]\mathrm{d}\xi\,\mathrm{d}\eta \\ & -\mathrm{i}\nu\frac{A^{-1}}{4\pi}\mathrm{FP}\int_{-\infty}^{0}\mathrm{e}^{\mathrm{i}\nu v}\,\mathrm{FP}\iint_S \frac{[\![\psi]\!]}{(y-\eta)^2}\left[1+\frac{A^{-1}(x-\xi)+v+x_0(y)-x_0(\eta)}{[(A^{-1}(x-\xi)+v+x_0(y)-x_0(\eta))^2+(y-\eta)^2]^{\frac{1}{2}}}\right] \\ & \times \mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}v. \end{aligned} \tag{13}$$

The first term of the right-hand side corresponds to the quasi-steady downwash. It is the velocity which would be induced on the wing if the jump of the acceleration potential $[\![\psi]\!]$, which occurs at time t, were imposed on S for an infinite duration. For further reference, this velocity is denoted by w_s.

The second term on the right-hand side of (13) is the purely unsteady downwash. It is denoted by w_u.

The interesting consequence of our introduction of non-dimensional variables is that the order of magnitude of the integration variables clearly appears. Hence, the term $A^{-1}(x-\xi)$ is of order A^{-1}, whereas terms like $x_0(y)-x_0(\eta)$ and $y-\eta$ are of unit order. It is clear now that asymptotic expansions of w_s and w_u may be found if the integrals are expanded with respect to the small variable $A^{-1}(x-\xi)$.

3.2. *Asymptotic expansion of* w_s

In a first step, an asymptotic expansion of the quasi-steady downwash w_s is sought. Actually, the procedure which has to be used is exactly the same as that which has been developed in Guermond (1990) to treat the steady case. In order to take advantage of existing tools which have been devised to find asymptotic expansions of line integrals, the chordwise and the spanwise integrations are carried out separately. The downwash w_s is put into the following form:

$$w_s = A^{-1}\,\mathrm{FP}\int_{-\infty}^{+\infty} I(\epsilon)\,\mathrm{d}\xi, \tag{14}$$

where ϵ is the small parameter $A^{-1}(x-\xi)$ and $I(\epsilon)$ is defined by

$$I(\epsilon) = \frac{1}{4\pi}\mathrm{FP}\int_{L(\xi)} \frac{\mathscr{H}[\![\psi]\!]}{(y-\eta)^2}\left[1+\frac{\epsilon+x_0(y)-x_0(\eta)}{[(\epsilon+x_0(y)-x_0(\eta))^2+(y-\eta)^2]^{\frac{1}{2}}}\right]\mathrm{d}\eta. \tag{15}$$

The notation $L(\xi)$ means that the integration is performed along a path where ξ is constant; these ξ-lines are parallel to L. The operator $\mathscr{H}$ is defined by

$$\mathscr{H}[\![\psi]\!](\xi,\eta) = [H(\xi-c_l(\eta))-H(\xi-c_t(\eta))]\,[\![\psi]\!](\xi,\eta). \tag{16}$$

Here, H is the classical Heaviside function. The operator $\mathscr{H}$ is introduced so that the pressure jump is implicitly zero off the wing. Hence, the problem consists of finding an asymptotic expansion of $I(\epsilon)$. The method for finding the asymptotic expansion in question with respect to the sequence $\{\epsilon^j \log(\epsilon)\}$ is given in Guermond (1990). At this point it is necessary to assume that the variation of the flow along the span occurs on the long lengthscale B. This assumption implies that $[\![\psi]\!]$ is smooth along the span and has derivatives with respect to y of order one. The asymptotic expansion of $I(\epsilon)$ is carried out up to $o(1)$, and the result is

$$I(\epsilon) = I_0(\xi)/\epsilon + I_1(\xi)\log|\epsilon| + I_2(\xi) + o(1), \tag{17}$$

where functions I_0, I_1, and I_2 are given by

$$I_0(\xi) = -\frac{\mathscr{H}[\![\psi]\!](\xi, y)}{2\pi\cos(\Lambda)}, \tag{18}$$

$$I_1(\xi) = \left[\frac{1}{4\pi r(y)} + \frac{\sin(\Lambda)}{2\pi}\frac{\partial}{\partial y}\right]\mathscr{H}[\![\psi]\!](\xi, y), \tag{19}$$

$$\begin{aligned} I_2(\xi) = {} & \frac{\mathscr{H}[\![\psi]\!](\xi, y)}{4\pi r(y)}\left[1 - \tan^2(\Lambda) - \log\left|\frac{2}{\cos^2(\Lambda)}\right|\right] \\ & + \frac{1}{2\pi}\frac{\partial}{\partial y}\mathscr{H}[\![\psi]\!](\xi, y)\left[\log\left|\frac{1+\sin(\Lambda)}{\cos(\Lambda)}\right| - \sin(\Lambda)\log\left|\frac{2}{\cos^2(\Lambda)}\right|\right] \\ & + \frac{1}{4\pi}\mathrm{FP}\int_{L(\xi)}\frac{\mathscr{H}[\![\psi]\!]}{(y-\eta)^2}\left[1 + \frac{x_0(y) - x_0(\eta)}{[(x_0(y) - x_0(\eta))^2 + (y-\eta)^2]^{\frac{1}{2}}}\right]\mathrm{d}\eta, \end{aligned} \tag{20}$$

where $r(y)$ is the local radius of curvature of the line L, and $\Lambda(y)$ is the angle between the local tangent of L at point $M_0(x_0(y), y, 0)$ and direction OY. The point M_0 is the projection of $M(x, y, 0)$ onto L along the OX-axis.

The physical interpretation of the expansion (17) will be given below. The final form of w_s will be obtained by integrating $I(\epsilon)$ with respect to ξ.

3.3. *Asymptotic expansion of w_u*

In the same spirit as in the previous section, the chordwise and spanwise integrations are separated as follows:

$$w_u = A^{-1}\,\mathrm{FP}\int_{-\infty}^{+\infty} J(\epsilon)\,\mathrm{d}\xi, \tag{21}$$

where $J(\epsilon)$ is given by

$$J(\epsilon) = -\mathrm{i}\nu\,\mathrm{e}^{-\mathrm{i}\nu\epsilon}\,\mathrm{FP}\int_{-\infty}^{\epsilon}\mathrm{e}^{\mathrm{i}\nu v}I(v)\,\mathrm{d}v. \tag{22}$$

At this stage, the problem is reduced to finding an asymptotic expansion of $J(\epsilon)$. In order to isolate a domain in which v is asymptotically small, the outer integral is divided into $\int_{-\infty}^{0}$ and $\int_0^{\epsilon}$. In the second integral v is necessarily $O(\epsilon)$; as a result, the asymptotic expansion of $I(\epsilon)$, (17), can be re-used replacing ϵ by v. After some manipulation, the asymptotic expansion of $J(\epsilon)$ is found up to $o(1)$:

$$\begin{aligned} J(\epsilon) = {} & -\mathrm{i}\nu\,\mathrm{e}^{-\mathrm{i}\nu\epsilon}I_0(\xi)\,\mathrm{FP}\int_{-\infty}^{\epsilon}\frac{\mathrm{e}^{\mathrm{i}\nu v}}{v}\mathrm{d}v \\ & -\mathrm{i}\nu\,\mathrm{e}^{-\mathrm{i}\nu\epsilon}\left[\int_{-\infty}^{0}\mathrm{e}^{\mathrm{i}\nu v}\left[I(v) - \frac{I_0(\xi)}{v}\right]\mathrm{d}v + I_1(\xi)\int_0^{\epsilon}\mathrm{e}^{\mathrm{i}\nu v}\log|v|\,\mathrm{d}v \right.\\ & \left. + I_2(\xi)\int_0^{\epsilon}\mathrm{e}^{\mathrm{i}\nu v}\,\mathrm{d}v\right] + o(1). \end{aligned} \tag{23}$$

Note that the term $-\mathrm{i}\nu\,\mathrm{e}^{-\mathrm{i}\nu\epsilon}I_0(\xi)\,\mathrm{FP}\int_{-\infty}^{0}\mathrm{e}^{\mathrm{i}\nu v}\,\mathrm{d}v/v$ has been added and subtracted in order to make a two-dimensional contribution appear. This operation will be physically interpreted below.

Up to this point, it has not been necessary to make any hypothesis on the order of magnitude of the reduced frequencies ν and k. As a consequence, the terms denoted by $o(1)$ are asymptotically negligible on the entire frequency domain when A increases to infinity. Now, the difficulty consists of finding a uniformly valid asymptotic arrangement of the terms which have been retained. This task is carried out in the next sub-section.

3.4. *Discussion on the asymptotic arrangement*

Before any consideration on the hierarchy of the terms, expressions (23) and (17) are summed. The sum of $I(\epsilon)$ and $J(\epsilon)$ is obtained after replacing ϵ by its value, $(x-\xi)/A$, and replacing v by τ/A in the integrals where ϵ was the upper bound:

$$
\begin{aligned}
I(\epsilon)+J(\epsilon) = {} & A I_0(\xi)\left[\frac{1}{x-\xi}-\mathrm{i}k\,\mathrm{e}^{-\mathrm{i}k(x-\xi)}\,\mathrm{FP}\int_{-\infty}^{x-\xi}\frac{\mathrm{e}^{\mathrm{i}k\tau}}{\tau}\,\mathrm{d}\tau\right] \\
& -\mathrm{i}\nu\,\mathrm{e}^{-\mathrm{i}k(x-\xi)}\int_{-\infty}^{0}\mathrm{e}^{\mathrm{i}\nu v}\left[I(v)-\frac{I_0(\xi)}{v}\right]\mathrm{d}v \\
& +I_1(\xi)\left[\log\left|\frac{x-\xi}{A}\right|-\mathrm{i}k\,\mathrm{e}^{-\mathrm{i}k(x-\xi)}\int_0^{x-\xi}\mathrm{e}^{\mathrm{i}k\tau}\log\left|\frac{\tau}{A}\right|\mathrm{d}\tau\right] \\
& +I_2(\xi)\left[1-\mathrm{i}k\,\mathrm{e}^{-\mathrm{i}k(x-\xi)}\int_0^{x-\xi}\mathrm{e}^{\mathrm{i}k\tau}\,\mathrm{d}\tau\right]+o(1).
\end{aligned} \tag{24}
$$

For the reader who may be interested in the calculations, it is worth noting that, for the integral defined in the finite-part sense, the variable change, $v = \tau/A$, is valid for every value of ϵ except zero. If ϵ is zero the equality is not satisfied unless a corrective term is added. As a result the equality is true almost everywhere on the chord of the wing except on the line $L(x)$. The surface of $L(x)$ being zero, the equality will be exact once the integration with respect to ξ is performed. For additional information on the rules of variable changing on finite-part integrals, the reader is referred to Lavoine (1959, 1963).

At this stage, some comments on the asymptotic arrangement of the right-hand side of (24) can be made. On the one hand, it is an easy task to verify that the first term of the right-hand side of (24) is asymptotically dominant throughout the entire frequency domain; it is of $O(A)$. This term is the two-dimensional contribution. On the other hand, the relative order of the corrections to the $O(A)$ leading term needs to be discussed. The corrections can be classified into three classes: the quasi-steady terms, the unsteady terms which involve the chord reduced frequency, and the unsteady terms which involve the span reduced frequency. It is clear that, in the low-frequency domain ($B = O(\lambda)$, i.e. $k = O(1/A)$ and $\nu = O(1)$), the leading corrections are of $O(\log(A))$ and $O(1)$. Among the unsteady contributions, only the term related to the span reduced-frequency scale, ν, is significant. As the frequency increases, the unsteady terms related to the chord reduced-frequency scale, k, become more significant, and the balance of the terms changes dramatically. In the Appendix it is shown that, in the high-frequency domain ($C = O(\lambda)$, i.e. $k = O(1)$ and $\nu = O(A)$), the quasi-steady terms are balanced by the unsteady ones, and the result is a term of order one, proportional to

$$\left(\frac{1}{4\pi r}+\frac{\sin(\Lambda)}{2\pi}\frac{\partial}{\partial y}\right)[\![\psi]\!].$$

As the frequency increases and reaches the very-high-frequency domain, the total correction finally becomes negligible. These results are summarized in table 1.

	Very low frequencies	Low frequencies $\nu = O(1)$	High frequencies $k = O(1)$	Very high frequencies
Quasi-steady terms	$O(\log(A))$ $O(1)$	$O(\log(A))$ $O(1)$	$O(\log(A))$ $O(1)$	$O(\log(A))$ $O(1)$
Terms related to ν	$o(1)$	$O(1)$	$O(\log(A))$ $O(1)$	$O(\log(A))$ $O(1)$
Terms related to k	$o(1)$	$o(1)$	$O(\log(A))$ $O(1)$	$O(\log(A))$ $O(1)$
Total	$O(\log(A))$ $O(1)$	$O(\log(A))$ $O(1)$	$O(1)$	$o(1)$

TABLE 1. Corrections to the leading terms in the asymptotic arrangement of equation (24)

In conclusion, the terms due to the chord reduced-frequency scale, the span reduced-frequency-scale, and the quasi-steady contribution are completely interwoven. Considering them separately would necessarily result in a theory which would not be uniformly valid. As a consequence, the last three terms of the right-hand side of (24) must be considered together.

3.5. *The asymptotic equation*

The asymptotic expansion of (13) is readily evaluated by integrating (24) with respect to ξ:

$$w(x,y) = \frac{1}{\cos(\Lambda)}\mathscr{K}_0[\![\psi]\!] + \frac{1}{A}\mathscr{K}_1[\![\psi]\!] + o\left(\frac{1}{A}\right), \tag{25}$$

where $\mathscr{K}_0$ and $\mathscr{K}_1$ are linear operators.

$\mathscr{K}_0$ is the classical two-dimensional operator:

$$\mathscr{K}_0[\![\psi]\!] = -\frac{1}{2\pi}\mathrm{FP}\int_{c_1(y)}^{c_t(y)}\left[\frac{1}{x-\xi} - \mathrm{i}k\,\mathrm{e}^{-\mathrm{i}k(x-\xi)}\,\mathrm{FP}\int_{-\infty}^{x-\xi}\frac{\mathrm{e}^{\mathrm{i}k\tau}}{\tau}\,\mathrm{d}\tau\right][\![\psi]\!](\xi,y)\,\mathrm{d}\xi. \tag{26}$$

Here, the finite-part integral reduces to a classical Cauchy principle value.

The operator $\mathscr{K}_1$ is defined by

$$\begin{aligned}\mathscr{K}_1[\![\psi]\!] = \mathrm{e}^{-\mathrm{i}kx}\Bigg\{&\left[\frac{1}{4\pi r(y)} + \frac{\sin(\Lambda)}{2\pi}\frac{\partial}{\partial y}\right]\left[G(y)\log\left(\frac{k}{\nu}\right) + H(x,y)\right]\\ &+\frac{G(y)}{4\pi r(y)}\left[1-\tan^2(\Lambda) - \log\left|\frac{2}{\cos^2(\Lambda)}\right|\right]\\ &+\frac{\dot{G}(y)}{2\pi}\left[\log\left|\frac{1+\sin(\Lambda)}{\cos(\Lambda)}\right| - \sin(\Lambda)\log\left|\frac{2}{\cos^2(\Lambda)}\right|\right]\\ &-\mathrm{i}\nu\frac{G(y)}{2\pi\cos(\Lambda)}\int_{-\infty}^{0}\frac{\mathrm{e}^{\mathrm{i}\nu v}}{v}\,\mathrm{d}v + w_0(M_0)\Bigg\},\end{aligned} \tag{27}$$

where functions $G(y)$ and $H(x,y)$ have the following definitions:

$$G(y) = \int_{c_1(y)}^{c_t(y)} \mathrm{e}^{\mathrm{i}k\xi}[\![\psi]\!](\xi,y)\,\mathrm{d}\xi, \tag{28}$$

$$H(x,y) = \int_{c_1(y)}^{c_t(y)}\left[\mathrm{e}^{\mathrm{i}kx}\log|x-\xi| - \mathrm{i}k\,\mathrm{e}^{\mathrm{i}k\xi}\int_0^{x-\xi}\mathrm{e}^{\mathrm{i}k\tau}\log|\tau|\,\mathrm{d}\tau\right][\![\psi]\!](\xi,y)\,\mathrm{d}\xi, \tag{29}$$

and the last term, $w_0(M_0)$, can be put into the form

$$w_0(M_0) = \frac{1}{4\pi}\mathrm{FP}\int_L \frac{G(\eta)}{(y-\eta)^2}\left[1+\frac{x_0(y)-x_0(\eta)}{[(x_0(y)-x_0(\eta))^2+(y-\eta)^2]^{\frac{1}{2}}}\right]\mathrm{d}\eta$$

$$-\frac{\mathrm{i}\nu}{4\pi}\mathrm{FP}\int_{-\infty}^{0} \mathrm{e}^{\mathrm{i}\nu v}\left[\mathrm{FP}\int_L \frac{G(\eta)}{(y-\eta)^2}\left[1+\frac{v+x_0(y)-x_0(\eta)}{[(v+x_0(y)-x_0(\eta))^2+(y-\eta)^2]^{\frac{1}{2}}}\right]\mathrm{d}\eta\right]\mathrm{d}v. \quad (30)$$

The dot above the G in (27) signifies a derivative with respect to y.

The calculations present no particular difficulties. The only technical point worth noting concerns the integration with respect to ξ of the term proportional to $(\partial/\partial y)\mathscr{H}[\![\psi]\!]$ in (20). The following relation is used:

$$\mathrm{FP}\int_{-\infty}^{+\infty}\frac{\partial}{\partial y}\mathscr{H}[\![\psi]\!](\xi,y)f(\xi)\,\mathrm{d}\xi = \frac{\partial}{\partial y}\mathrm{FP}\int_{c_1(y)}^{c_t(y)}[\![\psi]\!](\xi,y)f(\xi)\,\mathrm{d}\xi, \quad (31)$$

where $f(\xi)$ is any integrable function. The introduction of the operator $\mathscr{H}$ justifies the permutation of the integral sign and the derivative. It seems that this little point may have been overlooked in the pioneering work of Kida & Miyai (1978).

The logarithmic contributions have not been separated from the other since, as shown in §3.4, such a separation would not have been uniformly valid. Nevertheless, it is easy to verify that the asymptotic expansion is still of the Poincaré type with respect to the asymptotic sequence $\{1/A^j\}$. In other words, at each approximation order, J, the following equality holds uniformly on the entire frequency domain:

$$\lim_{A\to+\infty} A^J\left[w(x,y)-\sum_{j=0}^{J}\frac{\mathscr{K}_j(k,\nu)[\![\psi]\!]}{A^j}\right] = 0. \quad (32)$$

4. Physical interpretations

In order to interpret (25), the physical meaning of each term is sought, and possible links with the MAE approach are emphasized.

4.1. *Physical meaning of $G(y)$*

The velocity potential, ϕ, and the acceleration potential, ψ, are linked together by the operator, $\mathscr{B}$, and their respective jumps across the reference plane satisfy

$$[\![\phi]\!](\xi,\eta) = \mathrm{e}^{-\mathrm{i}k\xi}\int_{c_1(\eta)}^{\xi}\mathrm{e}^{\mathrm{i}k\tau}[\![\psi]\!](\tau,\eta)\,\mathrm{d}\tau \quad \text{if} \quad c_1 \leqslant \xi \leqslant c_t, \quad (33)$$

$$[\![\phi]\!](\xi,\eta) = \mathrm{e}^{-\mathrm{i}k(\xi-c_t(\eta))}\,\Gamma(\eta) \quad \text{if} \quad c_t \leqslant \xi, \quad (34)$$

where $\Gamma(\eta)$ is the circulation around the wing at the spanwise location η. The following equality is readily derived from the system (33), (34) and (28):

$$\Gamma(\eta) = \mathrm{e}^{-\mathrm{i}kc_t(\eta)}\,G(\eta). \quad (35)$$

Therefore, with the exception of the phase term, $\mathrm{e}^{-\mathrm{i}kc_t(\eta)}$, $G(\eta)$ represents the circulation.

It is also of interest to consider the vorticity distribution generated by the wing motion. Using Hess's (1972) theorem, the velocity induced by the potential jump, $[\![\phi]\!]$, may be shown to be the same as that induced by the vorticity distribution whose strength is $C(\boldsymbol{n}\wedge\nabla[\![\phi]\!])$, where $\boldsymbol{n}$ is the local normal vector. The vorticity distribution γ may be decomposed into $\gamma = \gamma_x+\gamma_y$, where γ_x is the component along the OX-axis and γ_y is the component along the tangent of the $L(\xi)$ line, which is inclined on the OY-axis with the angle $\Lambda(y)$. The magnitudes of vectors γ_x and γ_y are given, for bound vortices and free vortices, by

$$\gamma_x(\xi,\eta) = -\frac{e^{-ik\xi}}{A}\frac{\partial}{\partial\eta}\int_{c_l(\eta)}^{\xi} e^{ik\tau}[\![\psi]\!](\tau,\eta)\,d\tau \quad \text{if} \quad c_l \leqslant \xi \leqslant c_t, \tag{36}$$

$$\gamma_x(\xi,\eta) = -\frac{e^{-ik\xi}}{A}\dot{G}(\eta) \quad \text{if} \quad c_t \leqslant \xi, \tag{37}$$

$$\gamma_y(\xi,\eta) = -\frac{ik\,e^{-ik\xi}}{\cos(\Lambda)}\int_{c_l(\eta)}^{\xi} e^{ik\tau}[\![\psi]\!](\tau,\eta)\,d\tau + \frac{[\![\psi]\!]}{\cos(\Lambda)} \quad \text{if} \quad c_l \leqslant \xi \leqslant c_t, \tag{38}$$

$$\gamma_y(\xi,\eta) = -\frac{ik\,e^{-ik\xi}}{\cos(\Lambda)}G(\eta) \quad \text{if} \quad c_t \leqslant \xi. \tag{39}$$

4.2. *The domain decomposition*

Following the approach of the MAE technique, the influence of the vorticity distribution at the point M may be decomposed into that of the vortices located at distances of order C, and that of the vortices located at distances of order B or larger. In the reference plane, the inner domain, I, is defined as being the set of points whose distance from M is of order C. The diameter of I is set to an intermediate lengthscale, D, characterized by

$$C \ll D \ll B. \tag{40}$$

Two outer domains are defined. The first one, denoted by O_{wi}, is constituted of the points located in the wake of the inner domain at distances of order B or larger. The width of O_{wi} is set to the intermediate reference scale, D. The second outer domain, denoted by O, is composed of the points situated outside O_{wi} and at distances from M of order B or larger (see figure 2a). The velocity induced at M by the vorticity distribution is the sum of the contribution of the three domains, I, O_{wi}, and O.

4.3. *Velocity induced by domains O and O_{wi}*

In the domain $O \cup O_{wi}$, B is the reference lengthscale in both the spanwise and streamwise directions. As a result, from this domain, the details of the geometry of the wing in the chordwise direction become insignificant; the wing degenerates into the line L, and M merges into M_0 (see figure 2a). In this outer model, the line L is a lifting line as in Prandtl's model. Using the outer variable, v, the distribution of the jump of the velocity potential is given by

$$[\![\phi]\!](v,\eta) = e^{-i\nu v}G(\eta) \quad \text{for} \quad 0 \leqslant v. \tag{41}$$

Let $w_{out}(M_0)$ be the finite part of the downwash induced at M_0 by the vorticity distribution of the outer domains O and O_{wi}. It is obtained using the Green's representation theorem and differentiating the resulting velocity potential with respect to z:

$$w_{out}(M_0) = \frac{A^{-1}}{4\pi}\,\mathrm{FP}\int_L\int_0^{+\infty} \frac{[\![\phi]\!](v,\eta)}{[(v+x_0(\eta)-x_0(y))^2+(y-\eta)^2]^{\frac{3}{2}}}\,d\eta\,dv. \tag{42}$$

Substituting (41) into (42) and integrating by parts with respect to v leads to

$$w_{out}(M_0) = w_0(M_0). \tag{43}$$

This equation implies that $w_0(M_0)$ is the finite part of the downwash induced at M_0 by the lifting line whose shape is L and whose strength is $G(\eta)$.

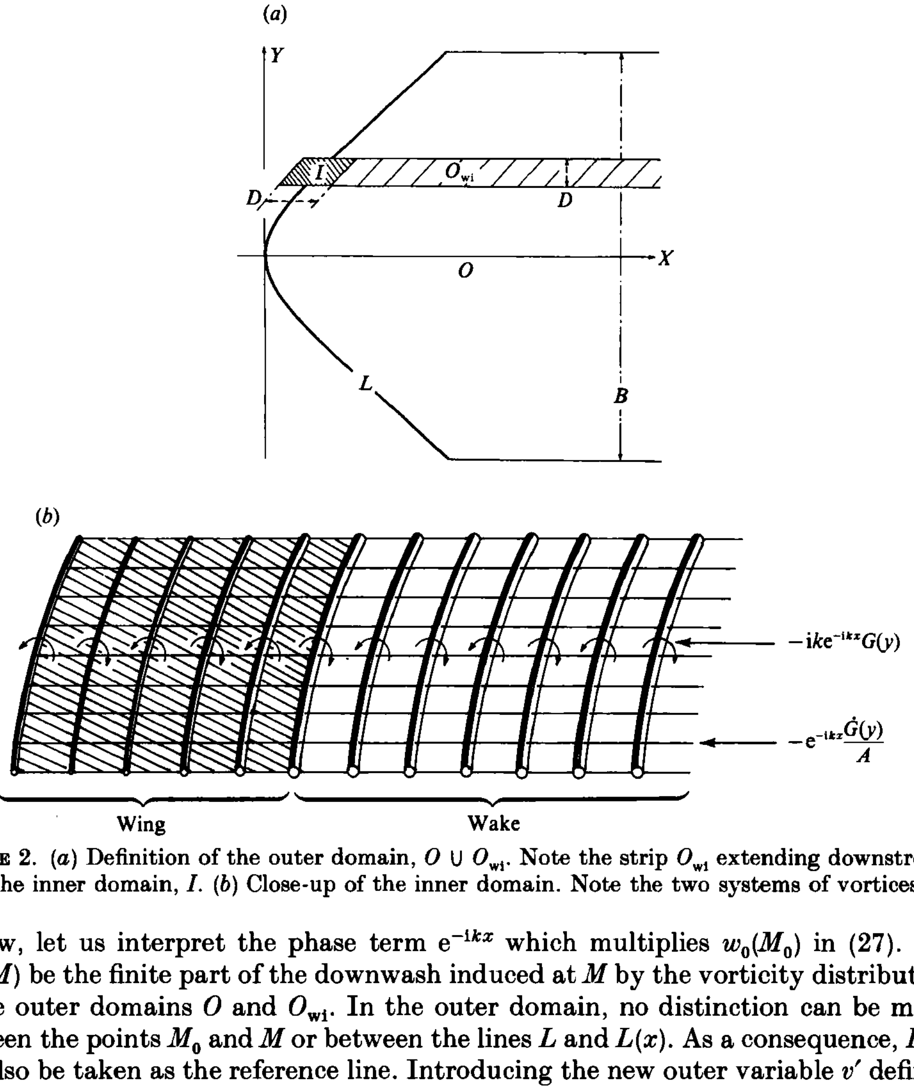

FIGURE 2. (a) Definition of the outer domain, $O \cup O_{wi}$. Note the strip O_{wi} extending downstream of the inner domain, I. (b) Close-up of the inner domain. Note the two systems of vortices.

Now, let us interpret the phase term e^{-ikx} which multiplies $w_0(M_0)$ in (27). Let $w_{out}(M)$ be the finite part of the downwash induced at M by the vorticity distribution of the outer domains O and O_{wi}. In the outer domain, no distinction can be made between the points M_0 and M or between the lines L and $L(x)$. As a consequence, $L(x)$ can also be taken as the reference line. Introducing the new outer variable v' defined by:

$$v' = v - x/A, \tag{44}$$

the jump of the velocity potential can be rewritten:

$$[\![\phi]\!]\,(v', \eta) = e^{-ikx} e^{-i\nu v'} G(\eta) \quad \text{for} \quad 0 \leqslant v', \tag{45}$$

where the exact condition, $-x/A \leqslant v'$, is replaced by its first approximation: $0 \leqslant v'$. Under this condition, the calculation of $w_{out}(M)$ is the same as that of $w_{out}(M_0)$ with the exception that the phase reference is shifted from L to $L(x)$:

$$w_{out}(M) = e^{-ikx} w_0(M_0). \tag{46}$$

The physical interpretation of $e^{-ikx} w_0(M_0)$ is now evident.

4.4. *Velocity induced by O_{wi}*

In the domain O_{wi}, the lengthscale in the spanwise direction is C, whereas B is the lengthscale in the streamwise direction. The width of O_{wi} being of order D, at the first approximation order, O_{wi} may be considered as a semi-infinite vortex sheet, whose straight and inclined boundary matches the local tangent of L at M_0, and whose vorticity strength γ_y, is $-\mathrm{i}\nu\, \mathrm{e}^{-\mathrm{i}\nu v} G(y)/A\cos(\Lambda)$. Note that the vortices are aligned with the support lines, $L(v)$. The downwash, $w_{\mathrm{wi}}(M_0)$, induced at M_0 by such a vortex distribution is readily evaluated:

$$w_{\mathrm{wi}}(M_0) = \frac{\mathrm{i}\nu}{A}\frac{G(y)}{2\pi\cos(\Lambda)}\int_{-\infty}^{0}\frac{\mathrm{e}^{\mathrm{i}\nu v}}{v}\mathrm{d}v. \tag{47}$$

Using the same arguments as for $w_{\mathrm{out}}(M)$, and changing the phase reference from L to $L(x)$, the downwash induced at M is deduced from that at M_0 by

$$w_{\mathrm{wi}}(M) = \mathrm{e}^{-\mathrm{i}kx} w_{\mathrm{wi}}(M_0). \tag{48}$$

This term is very important. It means that spanwise flow perturbations of lengthscale of order C, which are generated in the inner domain, I, by the wing motion are convected in the outer wake and are still active when they reach distances of order B and greater. Thus, in O_{wi}, the inner lengthscale, C, has a role as important as that of B. This is the double scale phenomenon which was recognized by Guiraud & Slama (1981). The domain O_{wi} plays the same role as that of Guiraud & Slama's narrow sheath surrounding the wake.

4.5. *Velocity induced by the inner domain*

In the inner domain, C is the common reference scale for the streamwise and spanwise directions, and the diameter of I is of order D. This domain is composed of the wing section on which M is situated together with its close wake (see figure 2*b* for a close-up of I).

At the first approximation order, only the γ_y-component of the vortices is of interest (see (38) and (39)). Furthermore, the vortices may be considered of constant strength in the spanwise direction, and the $L(\xi)$-lines can be approximated by inclined, straight lines parallel to the local tangent. The downwash induced by such vortex lines has a simple expression given by the Biot–Savart law, and it may be decomposed into the downwash induced by the bound vortices, $w_{0\mathrm{bound}}(M)$, and that induced by the free vortices travelling in the wake, $w_{0\mathrm{free}}(M)$:

$$w_{0\mathrm{bound}}(M) = -\frac{1}{2\pi\cos(\Lambda)}\int_{c_l(y)}^{c_t(y)}\frac{1}{x-\xi}\left[[\![\psi]\!](\xi,y) - \mathrm{i}k\,\mathrm{e}^{-\mathrm{i}k\xi}\int_{c_l(y)}^{\xi}\mathrm{e}^{\mathrm{i}k\tau}[\![\psi]\!](\tau,y)\,\mathrm{d}\tau\right]\mathrm{d}\xi, \tag{49}$$

$$w_{0\mathrm{free}}(M) = \frac{\mathrm{i}kG(\eta)}{2\pi\cos(\Lambda)}\int_{c_t(y)}^{\frac{D}{C}}\frac{\mathrm{e}^{-\mathrm{i}k\xi}}{x-\xi}\mathrm{d}\xi. \tag{50}$$

It is clear that $w_{0\mathrm{bound}}(M)$ and $w_{0\mathrm{free}}(M)$ are two-dimensional contributions, but their sum is not exactly the two-dimensional downwash, $w_{0\mathrm{in}}(M)$, because, however large the ratio D/C is, $w_{0\mathrm{free}}(M)$ will never include the influence of the vortices which have been convected out of I at distances of order B or larger. The vortices in question are those of O_{wi}; and the downwash they induce at M is $w_{\mathrm{wi}}(M)$. This velocity may be put into the alternative form:

$$w_{\mathrm{wi}}(M) = \frac{\mathrm{i}kG(y)}{2\pi\cos(\Lambda)}\int_{\frac{D}{C}}^{+\infty}\frac{\mathrm{e}^{-\mathrm{i}k\xi}}{x-\xi}\mathrm{d}\xi. \tag{51}$$

As a consequence, the exact two-dimensional downwash, $w_{0\mathrm{in}}(M)$, is the sum of $w_{0\mathrm{bound}}(M)$, $w_{0\mathrm{free}}(M)$, and $w_{\mathrm{wi}}(M)$. Hence, the physical interpretation of operator $\mathscr{K}_0$ is given by

$$\frac{\mathscr{K}_0[\![\psi]\!]}{\cos(\Lambda)} = w_{0\mathrm{bound}}(M) + w_{0\mathrm{free}}(M) + w_{\mathrm{wi}}(M) \equiv w_{0\mathrm{in}}(M). \tag{52}$$

At the second order of approximation, the curvature of $L(\xi)$-lines, together with the γ_x-component of the vortices (see (36) and (37)), has to be taken into account. It may be shown that the leading term of the asymptotic expansion of the downwash induced by the γ_x vortices and the curvature of γ_y vortices is

$$w_{1\text{in}}(M) = \frac{e^{-ikx}}{A}\left\{\left[\frac{1}{4\pi r(y)}+\frac{\sin(\Lambda)}{2\pi}\frac{\partial}{\partial y}\right]\left[G(y)\log\left(\frac{k}{\nu}\right)+H(x,y)\right]\right.$$
$$+\frac{G(y)}{4\pi r(y)}\left[1-\tan^2(\Lambda)-\log\left|\frac{2}{\cos^2(\Lambda)}\right|\right]$$
$$\left.+\frac{\dot{G}(y)}{2\pi}\left[\log\left|\frac{1+\sin(\Lambda)}{\cos(\Lambda)}\right|-\sin(\Lambda)\log\left|\frac{2}{\cos^2(\Lambda)}\right|\right]\right\}. \tag{53}$$

In conclusion the normal velocity induced by the inner and outer vortex systems is given by the sum

$$w(M) = w_{0\text{in}}(M)+w_{1\text{in}}(M)+w_{\text{out}}(M)-w_{\text{wi}}(M)+o(1/A). \tag{54}$$

The last term $w_{\text{wi}}(M)$ has to be subtracted once, since it is already taken into account in $w_{0\text{in}}$ and $w_{\text{out}}(M)$ (see §3.3). As was our intention, each term on the right-hand side of (25) has been given its physical interpretation.

5. Asymptotic solution

In this section, it is shown how (25) can be asymptotically solved, and some fundamental results are derived.

5.1. *A triangular system*

Even though the asymptotic expansion of (13) has been carried out only up to $o(1)$, in principle there is no limit to obtaining higher orders. Using the systematic technique developed in Guermond (1987, 1988, 1990), the asymptotic expansion of (13) with respect to the sequence $\{1/A^j\}$ can be obtained, with some algebraic effort, up to any order. Let $\sum_j^J a_j \mathscr{K}_j[\![\psi]\!]/A^j$ be such an expansion, where a_0 is $1/\cos(\Lambda)$ and every a_j is equal to 1 if j is not zero. Then, it is clear that the jump of the acceleration potential can be expanded with respect to the $\{1/A^l\}$ sequence. Re-introducing the asymptotic expansion of $[\![\psi]\!]$ into that of (13) yields a set of $J+1$ equations:

$$\mathscr{K}_0[\![\psi_0]\!] = \cos(\Lambda)\, w(M), \tag{55}$$

$$\mathscr{K}_0[\![\psi_j]\!] = -\cos(\Lambda)\sum_{\substack{i+l=j\\ i\neq 0}}\mathscr{K}_i[\![\psi_l]\!] \quad \text{for} \quad 1\leqslant j\leqslant J, \tag{56}$$

with the corresponding Kutta conditions at the trailing edge:

$$[\![\psi_j]\!] = 0 \quad \text{for} \quad \xi = c_t(\eta), \quad 0\leqslant j\leqslant J. \tag{57}$$

These conditions are of the utmost importance, for they permit unique solutions of problems (55) and (56) to be found.

Each equation of system (55), (56) is of two-dimensional type. Inverting operator $\mathscr{K}_0$ is a classical problem and presents no difficulties, see Ashley & Landhal (1965, p. 254) for details. The system is triangular; in other words, at each approximation order the right-hand side depends only on the previous orders. Therefore, the system can be progressively solved from the top to the bottom.

5.2. *Comments on the induced downwash*

Let $\mathscr{K}_0^{-1}$ be the inverse operator of $\mathscr{K}_0$ and $[\![\psi_0]\!]$ be the solution of (55), (57). In the present case, the jump of the acceleration potential is given, up to $o(1/A)$, by

$$[\![\psi]\!] = \cos(\Lambda)\,\mathscr{K}_0^{-1}\left[w(M) - \frac{1}{A}\mathscr{K}_1[\![\psi_0]\!]\right] + o(1/A). \tag{58}$$

Under this form, the downwash induced by the three-dimensional effects is readily recognized as being $-\cos(\Lambda)\,\mathscr{K}_1[\![\psi_0]\!]/A$. If the mean line, L, is straight and perpendicular to the stream, it is clear from (27) that the downwash has a sinusoidal dependence on x, the wavelength being $2\pi/k$. This result generalizes that of Ahmadi & Widnall (1985) for the entire frequency domain. This sinusoidal dependence is altered through the influence of the function $H(M)$ as curvature and inclination of the wing are to be accounted for.

5.3. *Self-averaging and double-scale effects*

In order to understand the influence of the frequency on the three-dimensional corrections, it is worth studying the degenerations of the present theory with respect to the frequency.

From table 1 it may be verified that in the low- and very-low-frequency domains two corrections of slightly different orders appear. One is $O(1/A)$ whereas the other is $O(\log(A)/A)$. This result is quite classical and needs no further comments.

As the frequency increases and reaches the high-frequency domain the three downwashes, $w_{1\mathrm{in}}$, w_{out}, and $-w_{\mathrm{wi}}$, compete (see the Appendix) and their combination yields a downwash proportional to the local curvature and sweep angle:

$$w_{1\mathrm{in}} + w_{\mathrm{out}} - w_{\mathrm{wi}} = \frac{\mathrm{e}^{-\mathrm{i}kx}}{A}\left[\frac{1}{4\pi r(y)} + \frac{\sin(\Lambda)}{2\pi}\frac{\partial}{\partial y}\right] \int_{c_1(y)}^{c_t(y)} \mathrm{e}^{\mathrm{i}k\xi}[\![\psi_0]\!](\xi, y)\left(\mathrm{FP}\int_{-\infty}^{x-\xi}\frac{\mathrm{e}^{\mathrm{i}k\tau}}{\tau}\mathrm{d}\tau\right)\mathrm{d}\xi + o\left(\frac{1}{A}\right). \tag{59}$$

The three-dimensional influence which is characterized by integrals involving the spanwise direction have been suppressed. This cancellation phenomenon corresponds to what was previously recognized by Cheng (1976) as the self-averaging effect. Actually, this effect is a consequence of the double-scale phenomenon in the sense that it is a consequence of the persistence of the small lengthscale, C, at distances of order B and larger. Furthermore, since w_{out} is the finite part of the velocity induced by both domains O and O_{wi}, (59) means that, in the high-frequency domain, the influence of the vortices of the domain O is asymptotically negligible, and the contribution of the two outer domains reduces to the finite part of the downwash induced by the narrow strip O_{wi}.

Note that if the wing is straight and unswept the term $w_{1\mathrm{in}}$ is zero and (59) reduces to $w_{\mathrm{out}} - w_{\mathrm{wi}} = o(1/A)$. In these conditions the finite-span correction is of $o(1/A)$; actually it may be shown to be of $O(\log(A)/A^2)$.

When the wing is curved or swept the right-hand side of (59) is not zero, and the finite-span correction is somewhat stronger than that in the previous case; it is of $O(1/A)$. The order of magnitude of this correction is higher than it was expected by some authors.

In the very-high-frequency domain, $A = o(k)$, the induced downwash (59) becomes of $O(\log(A)/A^2)$ whatever the geometry of the wing.

5.4. *The problem of the mean line position*

A question which may be raised about the present method concerns the mean line location. Where should this line be located and what is the influence of its position on the asymptotic solution?

The first part of the question has been answered in §2.1. In order to answer the second part, let F be a physical quantity, related to our problem, for which an asymptotic expansion is sought. Let F_1^J and F_2^J be two asymptotic expansions of F, up to order J, corresponding to two different mean line locations, L_1 and L_2:

$$F_1^J = \sum_{j=0}^{J} \frac{F_{1j}(k,\nu)}{A^j}, \quad F_2^J = \sum_{j=0}^{J} \frac{F_{2j}(k,\nu)}{A^j}. \tag{60}$$

F_1^J and F_2^J are related to F by

$$F = F_1^J + o(1/A^J), \quad F = F_2^J + o(1/A^J). \tag{61}$$

Subtracting the second equation from the first one shows that F_2^J differs from F_1^J by terms of $o(1/A^J)$. This means that expansions F_1^J and F_2^J are equal in the asymptotic sense. In other words, if the line L satisfies the hypotheses of §2.1, the asymptotic solution will not depend, in the asymptotic sense, on a particular choice of L.

5.5. *Limitations of the lifting-line model*

There are two inherent limitations to the lifting-line model. The first is that the wing tips must be cusped in order to allow slow variation of $[\![\psi]\!]$ in the tip regions. If this condition is not satisfied, the present asymptotic expansion is not uniformly valid in the vicinity of the wing tips. The diameter of the region in question is of $O(C)$. This restriction is classical and has been extensively discussed by Van Dyke (1964).

The second limitation concerns the predominance of the two-dimensional problem (55). In certain conditions the variation of the flow along the span may be so rapid (on the small scale C) that the lifting-line concept may not be relevant. In order to illustrate this phenomenon, consider a high-aspect-ratio wing operating under sinusoidal gust conditions. The imposed normal velocity takes the form

$$w(x,y) = w_0\, \mathrm{e}^{-\mathrm{i}\nu x_0(y)}\, \mathrm{e}^{\mathrm{i}kx}. \tag{62}$$

Using the fact that the operator $\mathscr{K}_0^{-1}$ is linear and does not operate on the variable y, from (55) we obtain

$$[\![\psi_0]\!] = w_0 \cos(\Lambda)\, \mathrm{e}^{-\mathrm{i}\nu x_0(y)} \mathscr{K}_0^{-1}[\mathrm{e}^{\mathrm{i}kx}]. \tag{63}$$

As a result, owing to the derivatives $\partial/\partial y$ and $\mathrm{d}/\mathrm{d}y$ in (27), the induced downwash, $-\cos(\Lambda)\,\mathscr{K}_1[\![\psi_0]\!]/A$, contains terms proportional to $\nu\dot{x}_0(y)/A$. The necessary condition for the asymptotic expansion (58) to be correctly arranged is that $\nu\dot{x}_0(y)$ must be of $o(A)$. As a result, if the wing is straight and unswept, that is $\dot{x}_0(y) = 0$, there is no condition imposed on the order of magnitude of ν, and the present theory is uniformly valid throughout the entire frequency domain. If the wing is curved or swept, that is $\dot{x}_0(y) = O(1)$, then ν must be of $o(A)$. This means that if the frequency is high or very high, the two-dimensional solution is not dominant. As a result, in this frequency domain the lifting-line concept is not relevant. Actually, in the frequency domain in question, the correct two-dimensional problem corresponds to that of an infinite swept wing with a constant chordlength in a sinusoidal gust. In this case, the variation of the flow occurs on the wavelength scale, λ, of the incoming sinusoidal gust.

In conclusion, this example shows that the present theory is consistent only if it satisfies the lifting-line axiom: $[\![\psi_0]\!]$ must vary along the span on the lengthscale B. This condition is uniformly satisfied in terms of frequency and wing geometry, if the wing undergoes heaving or pitching motions, whose amplitude may vary along the span on the scale B.

6. Lift and moment

In order to illustrate the present method, the lift and the nose-down pitching moment per unit span are calculated explicitly up to $o(1/A)$. The moment is taken about an axis along the midchord line. The non-dimensional lift, $l(y)$, and moment, $m(y)$, are introduced:

$$l(y) = \frac{L(y)}{\frac{1}{2}\rho U^2 C}, \quad m(y) = \frac{M(y)}{\frac{1}{2}\rho U^2 C^2}. \tag{64}$$

6.1. *Two-dimensional contribution*

The first task consists of evaluating the two-dimensional components. This calculation is classical. In order to facilitate the presentation, the lift operator, $\mathscr{L}_0$, is introduced:

$$\mathscr{L}_0(f) = -4C(k_1)\int_{c_1}^{c_t}\left(\frac{\xi-c_1}{c_t-\xi}\right)^{\frac{1}{2}} f(\xi)\,d\xi - \frac{8ik_1}{c(y)}\int_{c_1}^{c_t}[(\xi-c_1)(c_t-\xi)]^{\frac{1}{2}} f(\xi)\,d\xi, \tag{65}$$

where $f(\xi)$ represents the imposed normal velocity. The parameter k_1 is the local reduced frequency and is defined by $k_1 = \frac{1}{2}kc(y)$. Furthermore, $C(k_1)$ is Theodorsen's function (Theodorsen 1935):

$$C(k_1) = \frac{H_1^{(2)}(k_1)}{H_1^{(2)}(k_1) + iH_0^{(2)}(k_1)}. \tag{66}$$

$H_n^{(2)}(k_1)$ is the classical Hankel function of the second kind of order n. With these definitions, the two-dimensional lift, $l_0(y)$, is equal to $\mathscr{L}_0(w)$.

Likewise, it is convenient to define the moment operator, $\mathscr{M}_0$:

$$\begin{aligned}\mathscr{M}_0(f) = &-c(y)\,C(k_1)\int_{c_1}^{c_t}\left(\frac{\xi-c_1}{c_t-\xi}\right)^{\frac{1}{2}} f(\xi)\,d\xi \\ &+ c(y)\int_{c_1}^{c_t}\left[\left(\frac{\xi-c_1}{c_t-\xi}\right)^{\frac{1}{2}} - \frac{4}{c(y)}[(\xi-c_1)(c_t-\xi)]^{\frac{1}{2}}\right] f(\xi)\,d\xi \\ &+ ik_1 c(y)\int_{c_1}^{c_t}\left[\int_{c_1}^{\xi} f(\tau)\,d\tau\right]\left[\frac{1}{[(\xi-c_1)(c_t-\xi)]^{\frac{1}{2}}} - \frac{8}{c^2(y)}[(\xi-c_1)(c_t-\xi)]^{\frac{1}{2}}\right] d\xi.\end{aligned} \tag{67}$$

With this definition, the two-dimensional moment, $m_0(y)$, is equal to $\mathscr{M}_0(w)$.

6.2. *Three-dimensional corrections*

The total lift and moment are readily obtained from (58):

$$l(y) = \cos(\Lambda)\,\mathscr{L}_0(w(M) - 1/A\,\mathscr{K}_1[\![\psi_0]\!]) + o(1/A), \tag{68}$$

$$m(y) = \cos(\Lambda)\,\mathscr{M}_0(w(M) - 1/A\,\mathscr{K}_1[\![\psi_0]\!]) + o(1/A). \tag{69}$$

The calculation of the three-dimensional corrections presents no particular theoretical difficulty. As already mentioned in §5.2, the induced downwash is composed of a purely sinusoidal part and a remainder. See (27) for details. The purely sinusoidal part of the right-hand side of (27) is denoted by $w_{1s}(y)\,\mathrm{e}^{-\mathrm{i}kx}$ and the remaining part is denoted by $w_{1r}(x,y)$.

The lift, $l_{1s}(y)$, and moment, $m_{1s}(y)$, induced by the sinusoidal component of the downwash have a classical form which has been given by Sears (1941):

$$l_{1s}(y) = 2\pi c(y)\,\mathrm{e}^{-\mathrm{i}k_1(1-2K)}S(k_1)\,w_{1s}(y), \tag{70}$$

$$m_{1s}(y) = \pi\frac{c^2(y)}{2}\,\mathrm{e}^{-\mathrm{i}k_1(1-2K)}S(k_1)\,w_{1s}(y), \tag{71}$$

where K is the distance of M_0 from the leading edge, expressed as a fraction of the chord, $c(y)$. This parameter may be a function of the spanwise location, y. $S(k_1)$ is the Sears function, which has the following expression in terms of Theodorsen's function and Bessel functions of the first kind:

$$S(k_1) = C(k_1)\,[J_0(k_1)-\mathrm{i}J_1(k_1)]+\mathrm{i}J_1(k_1). \tag{72}$$

The lift, $l_{1r}(y)$, induced by the downwash $w_{1r}(x,y)$ is obtained after further calculations:

$$l_{1r}(y) = 2\pi c(y)\left\{\mathrm{e}^{-\mathrm{i}k_1(1-2K)}S(k_1)\,(\log(c(y))-\log(k_1)-\log(2)-\gamma-\tfrac{1}{2}\mathrm{i}\pi)\,\mathscr{A}G(y)\right.$$
$$\left.+\left(\frac{C(k_1)-1}{\mathrm{i}k_1}+1-2K\right)\mathscr{A}l_0(y)-\frac{2}{c(y)}\mathscr{A}l_0'(y)\right\}, \tag{73}$$

where the differential operator $\mathscr{A}$, defined by

$$\mathscr{A} = \left(\frac{1}{4\pi r(y)}+\frac{\sin(\varLambda)}{2\pi}\frac{\partial}{\partial y}\right), \tag{74}$$

has been introduced to render the formulae more readable; γ is Euler's constant; $l_0'(y)$ and $l_0''(y)$ are the first and second moments of the two-dimensional acceleration potential:

$$l_0'(y) = \int_{c_1(y)}^{c_t(y)} \xi[\![\psi_0]\!](\xi,y)\,\mathrm{d}\xi, \quad l_0''(y) = \int_{c_1(y)}^{c_t(y)} \xi^2[\![\psi_0]\!](\xi,y)\,\mathrm{d}\xi. \tag{75}$$

See Sellier's (1990) thesis for more details on the calculations.

The moment, $m_{1r}(y)$, is given by

$$m_{1r}(y) = \tfrac{1}{2}\pi c^2(y)\left\{\mathrm{e}^{-\mathrm{i}k_1(1-2K)}S(k_1)\,(\log(c(y))-\log(k_1)-\log(2)-\gamma-\tfrac{1}{2}\mathrm{i}\pi)\,\mathscr{A}G(y)\right.$$
$$\left.+\frac{1}{2}\left(\frac{C(k_1)-1}{\mathrm{i}k_1}+(1-2K)^2-\tfrac{1}{2}\right)\mathscr{A}l_0(y)+\frac{2(2K-1)}{c(y)}\mathscr{A}l_0'(y)+\frac{2}{c^2(y)}\mathscr{A}l_0''(y)\right\}. \tag{76}$$

6.3. *Final results*

The total lift and moment are finally given by

$$l(y) = \cos(\varLambda)\left[l_0(y)+\frac{l_{1s}(y)+l_{1r}(y)}{\varLambda}\right]+o(1/\varLambda), \tag{77}$$

$$m(y) = \cos(\varLambda)\left[m_0(y)+\frac{m_{1s}(y)+m_{1r}(y)}{\varLambda}\right]+o(1/\varLambda). \tag{78}$$

It is easy to verify that when k converges to zero, these formulae degenerate into those that have been obtained in the steady case, Guermond (1990). Ahmadi &

Widnall's results, obtained in the low-frequency domain for a straight unswept wing, can be recovered by expanding the present ones with respect to k with the hypothesis $k = O(1/A)$.

7. Numerical results

In order to test the present theory, a numerical program has been developed. Comparisons with other approaches are summarized below.

7.1. *Comparison with Ahmadi & Widnall's results*

Ahmadi & Widnall devised a lifting-line theory for unswept wings of large aspect ratio oscillating at low frequency. In order to illustrate their approach the authors considered an elliptic wing oscillating in pitch and heave. The authors defined the aspect ratio A_a as $(2B)^2/S_a$, where S_a is the wing planform area and B the semispan. The root semichord is $4B/\pi A_a$; in order to be consistent with our notation, let C denote this ratio.

The reduced frequency k is defined as $\omega C/U$. The magnitudes of the heave and pitch motions are $w = \frac{1}{2}\mathrm{i}k\xi_0$ and $w = \xi_1(1+\mathrm{i}kx)$ respectively. The lift and moment coefficients are defined by

$$C_{LH} = \frac{BC}{\frac{1}{2}\mathrm{i}k\xi_0 S_a}\int_{-1}^{1} l(y)\,\mathrm{d}y, \tag{79}$$

$$C_{MH} = \frac{BC}{\mathrm{i}k\xi_0 S_a}\int_{-1}^{1} m(y)\,\mathrm{d}y, \tag{80}$$

$$C_{LP} = \frac{BC}{\xi_1 S_a}\int_{-1}^{1} l(y)\,\mathrm{d}y, \tag{81}$$

$$C_{MP} = \frac{BC}{2\xi_1 S_a}\int_{-1}^{1} m(y)\,\mathrm{d}y, \tag{82}$$

where subscripts H and P denote heave and pitch respectively.

Figure 3 shows the total lift and moment coefficients as vector diagrams for a range of values of k for an elliptic wing whose aspect ratio, A_a, is equal to 16. As expected, the results obtained by the present theory (circles) are in full agreement with that of Ahmadi & Widnall (triangles) in the low-frequency domain. Furthermore, as the reduced frequency, k, becomes of order one, the results obtained from the present theory converge to the two-dimensional results (solid lines), whereas the results from the low-frequency theory diverge as noted by the authors.

Figure 4 shows the amplitude and phase of the induced downwash,

$$-\cos(\Lambda)\, w_{1\mathrm{s}}/A\xi_1$$

for an elliptic wing in pitch. The present theory is in agreement with the low-frequency theory when k is $o(1)$. The aspect ratio, A_a, is equal to 6. For the same reasons as stated above, some discrepancy is evident when the reduced frequency increases.

7.2. *Comparisons with a panel method*

In order to complete the series of numerical tests, we now compare the pressure jump distribution given by the theory with that given by a panel method that Cheng & Murillo (1984) used for testing their low-frequency theory. The numerical method was devised by Albano & Rodden (see Cheng & Murillo 1984 for further references).

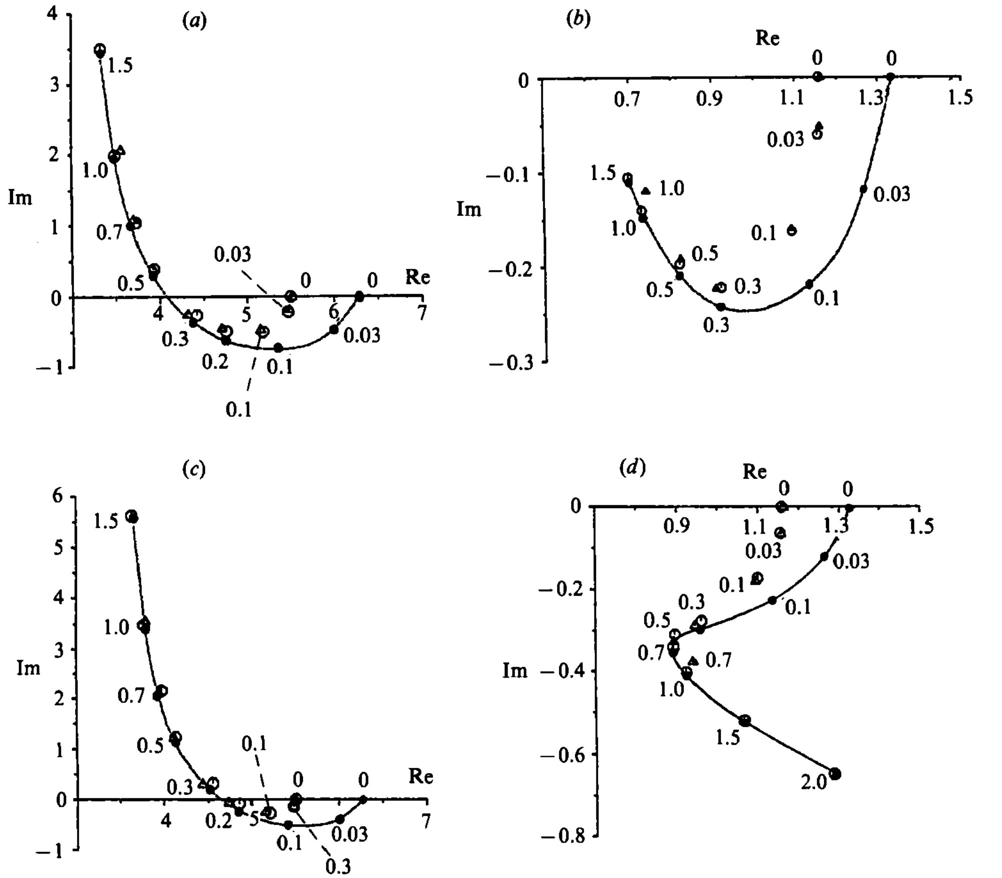

FIGURE 3. (a), (b) Complex vector diagram of $-C_{LH}$ and $-C_{MH}$ as functions of k for an elliptic wing in heave. (c), (d) Complex vector diagram of $-C_{LP}$ and $-C_{MP}$ for an elliptic wing in pitch. ($A_a = 16$); Re and Im respectively denote real and imaginary parts of the coefficients; —●—, 2D theory; △, low-frequency theory; ○, present theory.

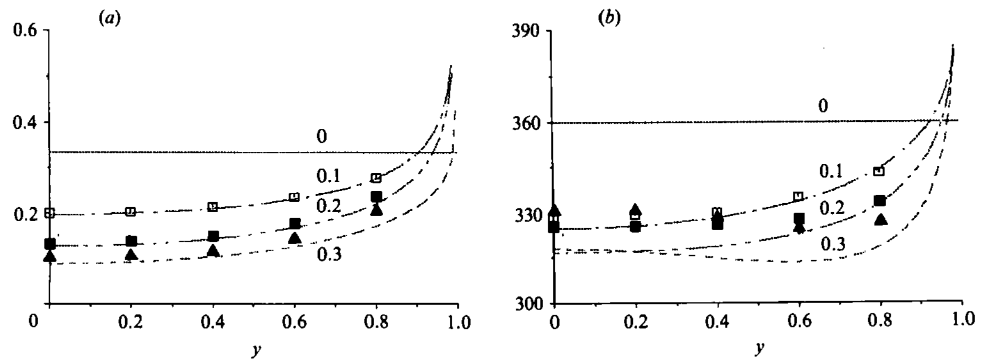

FIGURE 4. (a) Amplitude and (b) phase of the downwash for an elliptic wing in pitch ($A_a = 6$); lines represent result from the present theory, symbols represent results from the low-frequency theory: □, $k = 0.1$; ■, $k = 0.2$; ▲, $k = 0.3$.

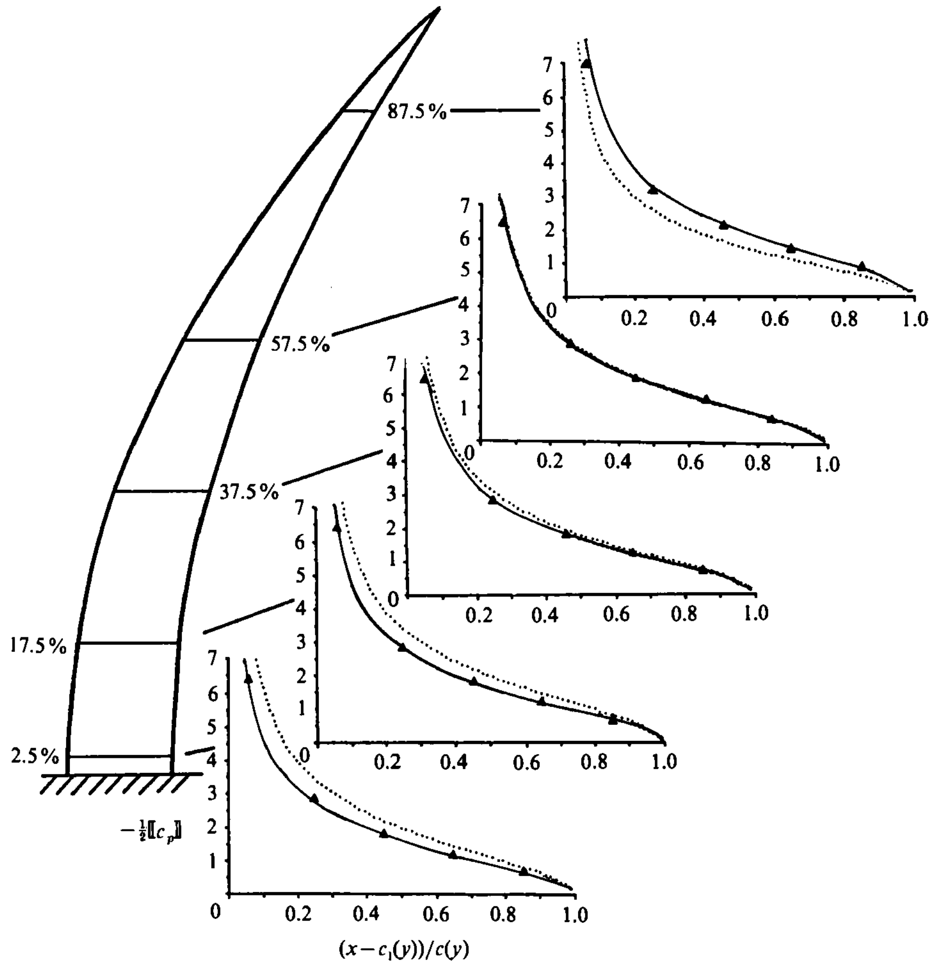

FIGURE 5. Chordwise pressure distribution on a parabolic fin in steady flow: ▲, the panel method; ······, the 2D approximation; ——, the present method.

The set of numerical results concerns a parabolic wing. The semispan and the root semichord are chosen as lengthscales B and C respectively. The wing planform is described by the set of equations:

$$K = 0, \quad x_0(y) = \tfrac{1}{2}y^2, \quad c(y) = 2(1-y^2)(1+y^2)^{\frac{1}{2}} \quad \text{for} \quad -1 \leqslant y \leqslant +1. \tag{83}$$

The ratio of B to C, still denoted by A, is equal to 15.

Figure 5 represents results for the chordwise pressure distribution, $-\frac{1}{2}[\![c_p]\!]$, versus $(x-c_1(y))/c(y)$ in the quasi-steady limit $(\nu \rightarrow 0)$. The flow tangency condition is expressed as $w = 1$. The pressure distribution determined from the present theory (solid lines) and from the panel method (triangles) are shown for five span locations: $y = 0.025$, 0.175, 0.375, 0.575, and 0.875. The consistent agreement of the present theory with the panel method is clear.

Figure 6 represents the chordwise pressure distribution, $-\frac{1}{2}[\![c_p]\!]$, for a pitching motion whose reduced frequency ν is equal to 1, and whose axis is located at $X_a \equiv Bx_a$, where $x_a = -0.2$. The flow tangency condition is expressed as $w = -1 - \mathrm{i}\nu(x_0(y) - x_a + x/A)$. Once more, the agreement of the asymptotic theory with the numerical method is clear.

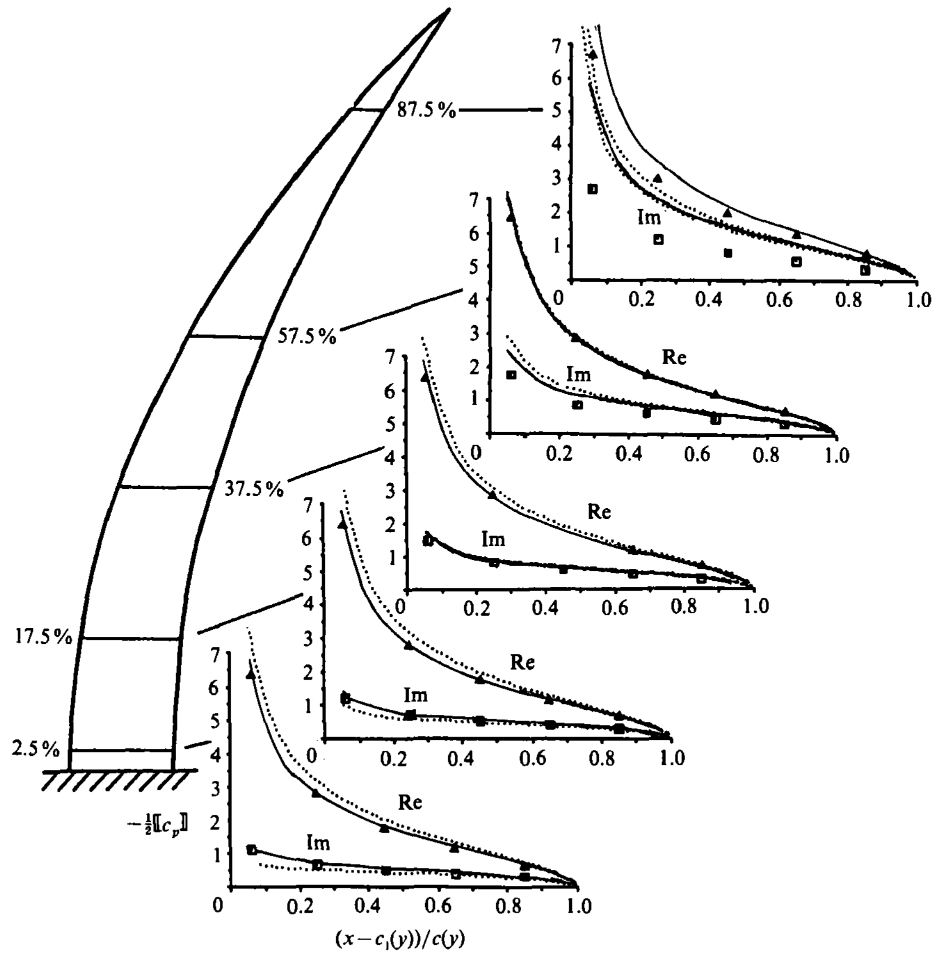

FIGURE 6. Chordwise pressure distribution on a parabolic fin in pitching oscillation: symbols, the panel method; ······, the 2D approximation; ——, the present method.

Figure 7 represents the chordwise pressure distribution, $-\frac{1}{2}[\![c_p]\!]$, for a feathering motion whose reduced frequency, ν, is equal to 1. This motion results from the combination of a primary heaving–pitching motion with the pitch axis at $X_a \equiv Bx_a$, and a secondary heaving–pitching motion whose amplitude depends on the span location, and whose pitch axis is set at the normal distance $X_b \equiv Cx_b$ from the centreline. This kind of motion models the fluttering of fish fins. See Cheng & Murillo for additional details on the animal propulsion aspect of this problem. For the present case, the imposed normal velocity takes the form:

$$w(x,y) = 1-\gamma+\mathrm{i}\nu[x_0(y)-x_a]+\mathrm{i}\nu/A[x(1-\gamma)+\gamma x_b(1+y^2)^{\frac{1}{2}}] - \frac{2\beta x}{A(1+y^2)}+\frac{x_b y^2}{A(1+y^2)^{\frac{3}{2}}}[\mathrm{i}\Theta(1+y^2)+\gamma], \quad (84)$$

where the coefficients γ and β are given by

$$\gamma = 1+\mathrm{i}\Theta[\tfrac{1}{2}y^2-x_a], \quad \beta = y^2\left[\mathrm{i}\Theta+\frac{\gamma}{1+y^2}\right]. \quad (85)$$

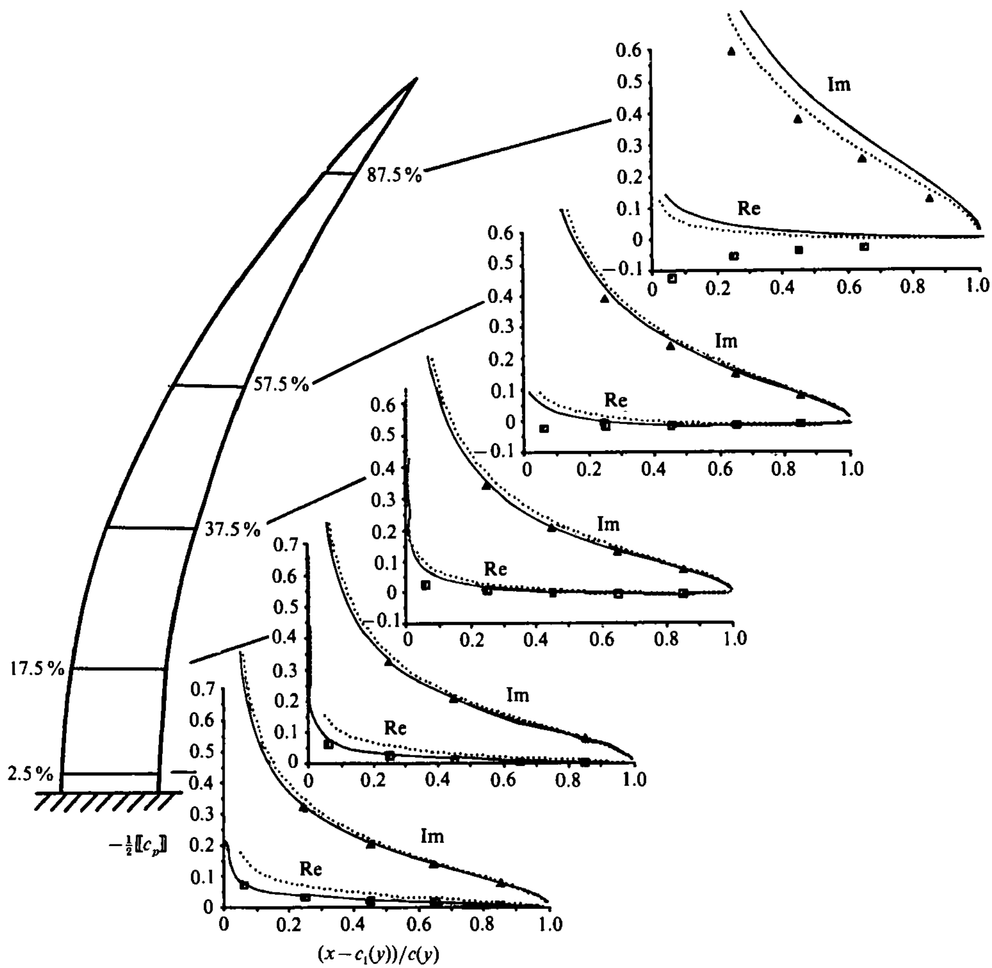

FIGURE 7. Chordwise pressure distribution on a parabolic fin in fluttering oscillation: symbols, the panel method; ······, the 2D approximation; ——, the present method.

Θ is the feathering parameter; it is set to 0.6 in the present case. Parameters x_a and x_b are set to -0.2 and 0.5.

On a large part of the wing the agreement of the present theory with the panel method is good. Some discrepancy, however, arises in the vicinity of the wing tip. It seems that this discrepancy might come from difficulties that Cheng & Murillo noted in prescribing the flow tangency boundary condition for the Albano–Rodden code. Note that (84) is an asymptotic expansion, up to $o(1/A)$, of the original condition which was expressed in curvilinear coordinates.

8. Conclusions

A unified asymptotic theory for wings of large aspect ratio in unsteady conditions has been presented. It has been shown that an asymptotic expansion which is uniformly valid on the entire range of frequency can be found if one does not try to *isolate logarithmic* contributions. As a result, the solution is expanded with respect to the sequence $\{1/A^j\}$ instead of the sequence $\{\log^i(A)/A^j\}$. The solution thus found has been shown to reproduce the results of other authors in the respective domains of validity of their theories.

The present solution is recurrent in the sense that higher approximation orders can be calculated with respect to previous orders through explicit recurrent formulae. As an illustration of the recurrence character of the theory, Sellier (1990) gave the second-order corrections for the unswept wing. Recurrence is a feature of regular perturbation problems. Even though the present problem is a singular perturbation problem as discovered by Van Dyke (1964), the singular character of the perturbation, $1/A$, appeared only once in the course of the demonstration of the general formula which has been used to carry out the asymptotic expansion of $I(\epsilon)$ in (17) (see Appendix A of Guermond 1990 for details of this formula).

It is likely that the present approach may be suitable for solving a larger class of integral equations for which a small parameter can be identified.

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Appendix

In this appendix, the asymptotic arrangement of $I(\epsilon)+J(\epsilon)$ is studied in the high- and very-high-frequency domains. Starting from (17) and (22), the following result is obtained:

$$I(\epsilon)+J(\epsilon)=\frac{I_0}{\epsilon}+I_1\log|\epsilon|+I_2-\mathrm{i}\nu\,\mathrm{e}^{-\mathrm{i}\nu\epsilon}\,\mathrm{FP}\int_{-\infty}^{\epsilon}\mathrm{e}^{\mathrm{i}\nu v}I(v)\,\mathrm{d}v. \tag{A 1}$$

This equality may be put into the alternative form:

$$\begin{aligned}I(\epsilon)+J(\epsilon)=I_0\left[\frac{1}{\epsilon}-\mathrm{i}\nu\,\mathrm{e}^{-\mathrm{i}\nu\epsilon}\,\mathrm{FP}\int_{-\infty}^{\epsilon}\frac{\mathrm{e}^{\mathrm{i}\nu v}}{v}\mathrm{d}v\right]\\+I_1\log|\epsilon|+I_2-\mathrm{i}\nu\,\mathrm{e}^{-\mathrm{i}\nu\epsilon}\,\mathrm{FP}\int_{-\infty}^{\epsilon}\mathrm{e}^{\mathrm{i}\nu v}\left[I(v)-\frac{I_0}{v}\right]\mathrm{d}v.\end{aligned} \tag{A 2}$$

The term proportional to I_0 is uniformly of order $1/A$. It is the leading term, the two-dimensional contribution. Let $R(\epsilon)$ be the remaining term. It may be rewritten

$$\begin{aligned}R(\epsilon)=I_1\log|\epsilon|+I_2-\mathrm{i}\nu\,\mathrm{e}^{-\mathrm{i}\nu\epsilon}\,\mathrm{FP}\int_{-\infty}^{\epsilon}\mathrm{e}^{\mathrm{i}\nu v}\left[I(v)-\frac{I_0}{v}-I_1\log|v|\right]\mathrm{d}v\\-\mathrm{i}\nu I_1\,\mathrm{e}^{-\mathrm{i}\nu\epsilon}\,\mathrm{FP}\int_{-\infty}^{\epsilon}\mathrm{e}^{\mathrm{i}\nu v}\log|v|\,\mathrm{d}v,\end{aligned} \tag{A 3}$$

where the finite-part integral has been regularized in the vicinity of ϵ so that the integrand behaves like I_2 when v is small. The $I_1\log|\epsilon|$ term disappears when the second integral is integrated by parts with respect to v:

$$R(\epsilon)=I_2-\mathrm{i}\nu\,\mathrm{e}^{-\mathrm{i}\nu\epsilon}\,\mathrm{FP}\int_{-\infty}^{\epsilon}\mathrm{e}^{\mathrm{i}\nu v}\left[I(v)-\frac{I_0}{v}-I_1\log|v|\right]\mathrm{d}v+I_1\,\mathrm{e}^{-\mathrm{i}\nu\epsilon}\,\mathrm{FP}\int_{-\infty}^{\epsilon}\frac{\mathrm{e}^{\mathrm{i}\nu v}}{v}\mathrm{d}v. \tag{A 4}$$

At this point, it is easy to verify that if the frequency is high or very high ($k = O(1)$, or $k \gg 1$) the first integral is equal to $I_2\,\mathrm{e}^{\mathrm{i}\nu\epsilon}/\mathrm{i}\nu$ plus terms $o(1)$. As a result, in the high-frequency domain, $R(\epsilon)$ takes the form

$$R(\epsilon) = I_1\,\mathrm{e}^{-\mathrm{i}\nu\epsilon}\,\mathrm{FP}\int_{-\infty}^{\epsilon}\frac{\mathrm{e}^{\mathrm{i}\nu v}}{v}\,\mathrm{d}v + o(1). \tag{A 5}$$

The leading term is easily seen to be at most of order one in the high-frequency domain when ϵ is replaced by its value and v is changed by τ/A. In the very high-frequency domain $R(\epsilon)$ becomes negligible whatever the geometry of the wing.

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